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Asymptotic Integration of a System Resulting from the Perturbation of an h-System

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1. INTRODUCTION

h-systems were introduced in [10] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given by exponential stability) under some perturbations. In this paper, we get asymptotic formulae for these systems which state new results about asymptotic integration for nonlinear systems under general hypotheses. Moreover, the corresponding results for linear systems give new insight about Levinson's theorem on asymptotic integration. In this way using the h-systems we get a uniform treatment for the concept of stability and we verify that several classical theorems of stability take a precise form by the asymptotic formulae.

In Section 2, we recall the definition and some properties of an h-system and we state two new theorems about asymptotic behavior of solutions of perturbed h-systems.

In Section 3, some general results about asymptotic integration are given. These results extend those given in [10]. In Section 4, we apply the technique to the linear case. Finally, in Section 5 we show examples and applications and we study the Emden–Fowler equation which originally does not satisfy the general hypothesis but can be transformed into one where they are satisfied.

2. h-SYSTEMS

Let us consider the differential system

$$x' = f(t, x), \qquad f(t, 0) \equiv 0, \tag{1}$$

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where f is a C^1 function on $I_a \times D$, $I_a = [a, \infty)$ and $D \subset \mathbb{R}^m$ is a domain containing the origin.

Let us denote by $x(t, t_0, x_0)$ the solution of (1) passing through the point $(t_0, x_0) \in I_a \times D$. We suppose that $x(t, t_0, x_0)$ is defined for all $t \geqslant t_0$ and that there exists a function continuous and positive on I_a and a positive constant c such that for all $x_0 \in D$ small enough,

$$|x(t, t_0, x_0)| \leqslant c\, |x_0|\, h(t)\, h(t_0)^{-1}, \qquad t \geqslant t_0 \geqslant a. \tag{2}$$

The function h as well as the constant c depend only on f.

A system (1) with these properties will be called an *h-system around the null solution* or, briefly, an *h-system*. In [10], an h-system was called h-A.S., asymptotically stable with respect to h because, since we were aiming at stability, originally we imposed that $h(t)$ would tend to zero for $t \to \infty$. Moreover, as we shall see, continuity of h is enough to obtain results concerning the extension of the domain of solutions, existence of solutions on a half-axis, asymptotic integration, etc.

Throughout this work all functions to be considered are supposed to be continuous on their domain of definition.

Although the basic results on perturbation of h-systems may be found in [10], in this section we will cite in way of example, two of them, which do not appear in [10] but are a consequence of the results there.

First, we need to study a perturbed system of the form

$$y' = f(t, y) + g(t, y) \tag{3}$$

with a continuous function g: $I_a \times D \to \mathbb{R}^n$. By Alekseev's formula [13]:

$$y(t) := y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \phi(t, s, y(s))\, g(s, y(s))\, ds, \tag{4}$$

where

$$\phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0) \tag{5}$$

is the fundamental matrix of the variational system

$$z' = f_x(t, x(t, t_0, x_0))\, z \tag{6}$$

such that $\phi(t_0, t_0, x_0) = I$ (f_x stands for the derivative of f with respect to x).

In what follows, the basic hypothesis will be that (6), for small x_0, is an h-system. (6) being an h-system means (see [10])

$$|\phi(t, t_0, x_0)| \leqslant c h(t)\, h(t_0)^{-1} \qquad (t \geqslant t_0 \geqslant a). \tag{7}$$

We note that (6) is an h-system implies (1) is an h-system, because by (5) we have

$$x(t, t_0, x_0) = \left[\int_0^1 \phi(t, t_0, \lambda x_0)\, d\lambda \right] x_0.$$

With the help of the results of [10] we shall obtain Theorems 1 and 2. We shall only prove Theorem 2, since its proof, as the reader will see, contains the proof of Theorem 1.

THEOREM 1. *Let us suppose that*

(H) *The system* (6) *is an h-system for* x_0 *small enough.*

(H_1) *The continuous functions* g_1, g_2 *on* $I_a \times D$ *satisfy*

$$|g_1(t, y)| \leqslant \lambda(t)\, |y|, \qquad \lambda \in C(I_a) \tag{8}$$

$$|g_2(t, y)| \leqslant \varepsilon\, |y|, \tag{9}$$

for ε *a nonnegative constant,* $t \in I_a$, *and* y *in a neighborhood of the origin. Then*

(T_1) *For all* $t_0 \geqslant a$ *and* y_0 *small enough, all solutions* $y(t, t_0, y_0)$ *of the perturbed system*

$$y' = f(t, y) + g_1(t, y) + g_2(t, y) \tag{10}$$

are defined for all $t \geqslant t_0$.

(T_2) *System* (10) *is a* k_ε*-system, where*

$$k_\varepsilon(t) = h(t) \cdot \exp\left[\int_a^t (\lambda(s) + \varepsilon)\, ds \right].$$

(T_3) *All these solutions tend to zero when* $t \to \infty$ *if*

$$h(t) = o\left(\exp - \left[\varepsilon t + \int_a^t \lambda(s)\, ds \right] \right), \qquad t \to \infty.$$

We may still add the perturbation g_3, continuous on $I_a \times D$ such that for $t \in I_a$ and y in a neighborhood V of the origin, satisfies

$$|g_3(t, y)| \leqslant v(t), \qquad \int_t^{t+1} v(s)\, ds \to 0 \quad \text{as} \quad t \to \infty, \quad v \in C(I_a). \tag{11}$$

THEOREM 2. *Let us assume*

(H) *System* (6) *is an h-system for* x_0 *small enough.*

(H_1) *The function h fulfills the following conditions* (*see Remark* 1):

$$0 < \liminf_{t \to \infty} h(t) \int_t^{t+1} h(s)^{-1}\, ds \leqslant \limsup_{t \to \infty} h(t) \int_t^{t+1} h(s)^{-1}\, ds < \infty \tag{12}$$

$$\limsup_{t \to \infty} h(t) \int_a^t h(s)^{-1}\, ds = M \tag{13}$$

$$\liminf_{t \to \infty} h(t)\, e^{M^{-1}t} > 0. \tag{14}$$

(H_2) *The continuous functions* g_1, g_2, g_3 *satisfy respectively* (8), (9), *and* (11) *for* $t \in I_a$ *and* y *in a neighborhood* $V \subseteq D$ *of the origin.*

Then

(T_1) *For any* $t_0 \geqslant a$ *and* y_0 *small enough the solutions* $y(t, t_0, y_0)$ *of the perturbed system*

$$y' = f(t, y) + g_1(t, y) + g_2(t, y) + g_3(t, y) \tag{15}$$

are defined for all $t \geqslant t_0$.

(T_2) *If in* (9), $\varepsilon < M^{-1}$ *and in* (8), $\lambda \in L_1(I_a)$, *then for all* t_0 *big enough,* $y(t, t_0, y_0)$ *tends to zero when* $t \to \infty$.

Remark 1. Condition (13) (see Coppel [6, p. 68]) implies:

$$h(t) \leqslant N e^{-M^{-1}t}, \qquad t \geqslant a, \qquad N > 0 \quad \text{constant.} \tag{13'}$$

Then, (13) together with (14) ensures that the system is exponentially stable and (12) is automatically fulfilled. However, condition (14) is *only* needed if $\varepsilon \neq 0$. If $\varepsilon = 0$, i.e., if $g_2 \equiv 0$, (12) and (13) are sufficient to ensure the conclusions of the theorem. Moreover, if, for $t \to \infty$, $g_3(t, y) \to 0$ (without satisfying (11)) uniformly in $y \in V$, then to obtain the conclusion of Theorem 2 is enough to assume (13); (12) and (14) are not necessary. Finally, if $\varepsilon = 0$ and $\lambda \in L_1(I_a)$ the perturbed system (10) will be asymptotically stable as soon as h tends to zero for $t \to \infty$. In example 1 below, h-systems are shown which satisfy (12) and (13) but are not exponentially stable.

Proof of Theorem 2. Alekseev's formula (4) with $g = g_1 + g_2 + g_3$ establish that for $y(t) = y(t, t_0, y_0)$ we have

$$y(t) = x(t, t_0, y_0) + \int_{t_0}^{t} \phi(t, s, y(s))\, g_1(s, y(s))\, ds$$
$$+ \int_{t_0}^{t} \phi(t, s, y(s))\, g_2(s, y(s))\, ds$$
$$+ \int_{t_0}^{t} \phi(t, s, y(s))\, g_3(s, y(s))\, ds.$$

So, with the help of (2), (7), (8), (9), and (11), we have for y_0 small enough and t in a neighborhood of t_0 such that $y(t) = y(t, t_0, y_0) \in V$,

$$|y(t)| \leqslant c\,|y_0|\, h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} ch(t)\, h(s)^{-1}(\lambda(s) + \varepsilon)\,|y(s)|\, ds$$
$$+ \int_{t_0}^{t} ch(t)\, h(s)^{-1}\, v(s)\, ds$$

or

$$h(t)^{-1}\,|y(t)| \leqslant c\,|y_0| h(t_0)^{-1} + \int_{t_0}^{t} ch(s)^{-1}\, v(s)\, ds$$
$$+ \int_{t_0}^{t} ch(s)^{-1}(\lambda(s) + \varepsilon)\,|y(s)|\, ds.$$

Let $u(t)$ be the right-hand side of the last inequality. Since

$$c^{-1}u(t)' = h(t)^{-1}\, v(t) + h(t)^{-1}(\lambda(t) + \varepsilon)\,|y(t)|$$
$$\leqslant h(t)^{-1}\, v(t) + u(t)(\lambda(t) + \varepsilon),$$

then

$$\frac{d}{dt}\left(u(t) \exp\left[-c \int_{t_0}^{t} (\lambda(s) + \varepsilon)\, ds\right]\right) \leqslant \mathrm{ch}(t)^{-1}\, v(t) \exp\left[-c \int_{t_0}^{t} (\lambda(s) + \varepsilon)\, ds\right].$$

Integrating we get

$$u(t) \leqslant \exp c \int_{t_0}^{t} (\lambda(s) + \varepsilon)\, ds \left(|y_0|\, h(t_0)^{-1}\right.$$
$$\left. + c \int_{t_0}^{t} h(s)^{-1}\, v(s) \exp\left[-\int_{t_0}^{s} c(\lambda(\tau) + \varepsilon)\, d\tau\right] ds\right).$$

This means that

$$|y(t)| \leqslant c\,|y_0|\,h(t)\,h(t_0)^{-1} \exp \int_{t_0}^{t} c(\lambda(s)+\varepsilon)\,ds$$
$$+\, ch(t) \int_{t_0}^{t} h(s)^{-1}\,v(s) \exp\left[\int_{s}^{t} c(\lambda(\tau)+\varepsilon)\,d\tau\right] ds \tag{16}$$

which entails that the solution $y(t)$ is defined on the whole interval $[t_0, \infty)$.

Moreover, if $\varepsilon < M^{-1}$ and $\lambda \in L_1(I_a)$, it is immediate that the first summand of (16) tends to zero for $t \to \infty$. The proof that, for t_0 big enough, the second summand of (16) also tends to zero for $t \to \infty$, follows analogously to the proof of Theorem 2 [10], taking

$$h(t) \exp \int_{a}^{t} c(\lambda(s)+\varepsilon)\,ds$$

instead of h.

We remark, that taking $g_3 \equiv 0$, the proof of Theorem 2 contains that of Theorem 1. Moreover, the fact that (10) is a k_ε-system as ensured by Theorem 1, we suggest proving Theorem 2 by considering system (15) as a perturbed version of (10). The difficulty lies in that we need the variational systems corresponding to (10) be k_ε-systems too. In [10], it is seen that it is not easy, without further restrictions, to guarantee that the variational systems associated to an h-system also be h-systems. Our proof avoids all this and allows us to obtain these results for three simultaneous perturbations. Theorem 1 and 2 extend some results of [3].

Obviously, every function which is a multiple of a decreasing exponential function satisfy (12) and (13) of Theorem 2, but there are h-systems (1) such that h satisfies (12) and (13) but are not exponentially stable.

EXAMPLE 1. Consider the Ricatti scalar equation:

$$x' = \lambda(t)(-x+x^2). \tag{17}$$

The solutions are given by

$$x(t, t_0, x_0) = \left[1 + (x_0^{-1} - 1) \exp \int_{t_0}^{t} \lambda(s)\,ds\right]^{-1}$$

For $|x_0| < \frac{1}{2}$,

$$|x(t, t_0, x_0)| \leqslant \frac{|x_0|}{1-x_0} \exp\left[-\int_{t_0}^{t} \lambda(s)\,ds\right] \leqslant 2\,|x_0|\,h(t)\,h(t_0)^{-1},$$

where

$$h(t)=\exp\left[-\int_a^t \lambda(s)\,ds\right].$$

Therefore, (17) is a one-dimensional h-system. If we take b a positive and differentiable function in $[0, \infty)$ and

$$\lambda(t)=1+b'(t)/b(t)$$

then

$$h(t)=e^{-t}/b(t).$$

For each positive integer n let m_n and M_n be defined by

$$m_n=\min b([n, n+1]), \qquad M_n=\max b([n-\tfrac{1}{2}, n+\tfrac{1}{2}]), \qquad n\geqslant 1.$$

Suppose that $\{M_n\}$ and $\{m_n\}$ are strictly increasing sequences and for any $n\geqslant 2$ they satisfy the relations:

$$0<m_{n-1}<M_n; \qquad M_n/m_n\to\infty \qquad \text{as} \quad n\to\infty.$$

Under these conditions Eq. (17) is asymptotically stable but it is not exponentially stable. Moreover, for $t\leqslant s\leqslant t+1$ we have

$$b([t-1])\leqslant b(s)\leqslant b([t+2]),$$

where $[t]$ denotes the integer part of t. We also need the conditions

$$0<\lim_{n\to\infty}\frac{M_{n-1}}{M_{n+1}}\leqslant\lim_{n\to\infty}\frac{M_{n+2}}{M_{n-1}}<\infty,$$

which guaranty that (12) and (13) are satisfied. That is, (13) follows from

$$h(t)\int_a^t h(s)^{-1}\,ds=\frac{e^{-t}}{b(t)}\int_a^t e^s b(s)\,ds\leqslant\frac{b([t+1])}{b(t)}\,e^{-t}\int_a^t e^s\,ds$$

and (12) from

$$\begin{aligned}\frac{b([t-1])}{b(t)}\int_t^{t+1} e^{-(t-s)}\,ds&\leqslant\frac{e^{-t}}{b(t)}\int_t^{t+1} e^s b(s)\,ds\\&\leqslant\frac{b([t+2])}{b(t)}\int_t^{t+1} e^{-(t-s)}\,ds.\end{aligned}$$

3. Asymptotic Integration of Perturbed h-System

Our purpose is to obtain some results about asymptotic integration for systems of the type

$$y' = f(t, y) + g(t, y),$$

where

$$z' = f_x(t, x(t, t_0, x_0))\, z \tag{6}$$

is an h-system for x_0 small enough. First we need to solve an integral inequality. We use the class of function H as defined in Dannan [7].

Definition. A function w: $[0, \infty) \to [0, \infty)$ is said to belong to the class H if

(i) $w(u)$ is nondecreasing and continuous for $u \geqslant 0$ and positive for $u > 0$.

(ii) There exists a function (multiplier function) r, continuous on $[0, \infty)$ with $w(\alpha u) \leqslant r(\alpha)\, w(u)$ for $\alpha > 0$, $u \geqslant 0$.

Lemma 1. *Assume that $z(t)$ is a continuous nonnegative function on I_a, $w \in H$ with corresponding multiplier function r and $h(t) > 0$ is a continuous function on I_a such that*

$$\int_a^\infty \frac{\lambda(s)\, r(l(s, t_0))}{l(s, t_0)}\, ds < \infty,$$

where λ is a continuous nonnegative function on I_a and

$$l(t, t_0) = h(t)\, h(t_0)^{-1}, \qquad t \geqslant t_0 \geqslant a. \tag{18}$$

If

$$z(t) \leqslant c\, |y_0|\, h(t)\, h(t_0)^{-1} + \int_{t_0}^t ch(t)\, h(s)^{-1}\, \lambda(s)\, w(z(s))\, ds, \qquad t \geqslant t_0, \tag{19}$$

then, for any $t \geqslant t_0$,

$$z(t) \leqslant l(t, t_0)\, W^{-1}\left[W(c\, |y_0|) + \int_{t_0}^t \frac{c\lambda(s)\, r(l(s, t_0))}{l(s, t_0)}\, ds \right], \tag{20}$$

where

$$\begin{aligned} W(u) &= \int_{u_0}^u \frac{ds}{w(s)}, \qquad u > 0, \quad u_0 > 0, \\ W(0^+) &= -\infty, \end{aligned} \tag{21}$$

and W^{-1} is the inverse of W and $|y_0|$ is small enough such that in (20) W^{-1} *is defined.*

Proof. From (19) one obtains

$$\frac{z(t)}{h(t)\,h(t_0)^{-1}} \leqslant c\,|y_0| + \int_{t_0}^{t} \frac{c\lambda(s)\,w(z(s))}{h(s)\,h(t_0)^{-1}}\,ds.$$

Using (18) and the fact that $w \in H$, one has

$$\frac{z(t)}{l(t, t_0)} \leqslant c\,|y_0| + \int_{t_0}^{t} \frac{c\lambda(s)\,r(l(s, t_0))}{l(s, t_0)}\,w\left(\frac{z(s)}{l(s, t_0)}\right) ds.$$

Considering $z(t)/l(t, t_0)$ as a function, using Bihari's inequality [2], the result (20) follows. The condition (21) is necessary so that W^{-1} has meaning.

Using this lemma we can prove

THEOREM 3. *Suppose that* (H) *of Theorem* 1 *holds and so does*

(H_1) *The function h satisfies* (*see Remark* 2)

$$\lim_{t \to \infty} h(t) < \infty.$$

(H_2) *The contiuous function g for $(t, y) \in I_a \times D$ satisfies the inequality*

$$|g(t, y)| \leqslant \lambda(t)\,\omega(|y|),$$

where λ and ω are the same as Lemma 1.

Then

(T_1) *For any $t_0 \geqslant a$ and y_0 sufficiently small, every solution $y(t) = y(t, t_0, y_0)$ of the perturbed system*

$$y' = f(t, y) + g(t, y) \tag{3}$$

is defined for all $t \geqslant t_0$.

(T_2) *These solutions satisfy*

$$|y(t, t_0, y_0)| \leqslant Kh(t)\,h(t_0)^{-1},$$

where K is a positive constant.

(T_3) *For every solution as in* (T_1) y *of* (3) *there is a solution* x *of* (1) *such that, for* t *goes to infinity,*

$$y = x + h \cdot \tilde{o}(1),$$

where $\tilde{o}(1)$ *represents a function which has a limit when* t *approaches infinity.*

Proof. Using Alekseev's formula:

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Then by (2), (7), and (20) we have

$$|y(t, t_0, y_0)| \leqslant c\, |y_0|\, h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} ch(t)\, h(s)^{-1}\, \lambda(s)\, \omega(|y(s)|)\, ds.$$

Therefore by Lemma 1, for y_0 small enough and $t \geqslant t_0$, one has

$$|y(t, t_0, y_0)| \leqslant h(t)\, h(t_0)^{-1}\, W^{-1}\left[W(c\, |y_0|) + \int_{t_0}^{t} \frac{c\lambda(s)\, r(l(s, t_0))}{l(s, t_0)}\, ds \right]$$

and then, for y_0 small enough, there is a positive constant K such that

$$|y(t, t_0, y_0)| \leqslant Kh(t)\, h(t_0)^{-1}, \qquad t \geqslant t_0 \geqslant a,$$

In this way, without using (H_1), we obtain (T_1) and (T_2).

Now, for these solutions y of (3),

$$\begin{aligned} \left| \int_{t_0}^{t} \phi(t, s, y(s))\, g(s, y(s))\, ds \right| &\leqslant \int_{t_0}^{t} ch(t)\, h(s)^{-1}\, \lambda(s)\, \omega(|y(s)|)\, ds \\ &\leqslant ch(t)\, h(t_0)^{-1} \int_{t_0}^{t} \frac{\lambda(s)\, r(l(s, t_0)}{l(s, t_0)}\, \omega\left(\frac{|y(s)|}{l(s, t_0)} \right) ds \\ &\leqslant c\omega(K)\, h(t)\, h(t_0)^{-1} \int_{t_0}^{t} \frac{\lambda(s)\, r(l(s, t_0))}{l(s, t_0)}\, ds. \end{aligned}$$

So, for every solution y of (3), the solution x of (1) given by

$$x(t) = y(t) - \int_{t_0}^{t} \phi(t, s, y(s))\, g(s, y(s))\, ds$$

has the property

$$y(t) = x(t) + h \cdot \tilde{o}(1)$$

as t approaches infinity.

Remark 2. Condition (H_1) is not really necessary. Under the hypotheses (H) and (H_2) the asymptotic formula given in (T_3) is always true. However, if the function h is not a good mayorant, the fact that gets worse if (H_1) is not satisfied, all the information can be added to the error, if all we know about it is that it is smaller than h. That is the case in Euler's equation

$$y'' + (2t)^{-2} y = 0; \qquad x'' = 0.$$

A difficulty like this appears, for instance, with systems whose nonperturbed systems are linear with eigenvalues having a nonnegative real part and multiplicity greater than one. As we will see at the end of Section 5, where we study the Emden–Fowler equation, the constant case of only one eigenvalue is reduced, after some number of transformations to one whose linear part is uniformly stable and to this new system we apply our results. The case of more than one eigenvalue with multiplicity greater than one can be treated in a similar way (see [6, 9]).

Dannan [7] shows several examples of elements belonging to the class H. Also Theorem 3 can be applied to the case $w(u) = u^{n+1}$, $n \geq 0$, getting

THEOREM 4. *Suppose that condition* (H) *and* (H_1) *of Theorem* 3 *are fulfilled and*

(H_2) *There is a continuous function* λ *and* $n \geq 0$ *such that*

$$|g(t, y)| \leq \lambda(t)\, |y|^{n+1}, \qquad \lambda h^n \in L_1(I_a)$$

for all $(t, y) \in I_a \times D$.

Then (T_1) *and* (T_3) *hold and* (T_2) *becomes*

(T_2') *System* (3) *is an h-system.*

Proof. Since $\omega(u) = u^{n+1}$, $n \geq 0$ satisfies Theorem 3 and it only remains to prove that system (3) is an h-system. We know

$$|y(t, t_0, y_0)| \leq h(t)\, h(t_0)^{-1}\, W^{-1}\Big[W(c\,|y_0|) + [h(t_0)^{-1}]^n \int_{t_0}^{t} c\lambda(s)\, h^n(s)\, ds \Big],$$

for $|y_0|$ small enough. If $n = 0$, $W(u) = lnu$ and $W^{-1}(v) = e^v$ and

$$|y(t, t_0, y_0)| \leq c\, |y_0|\, h(t)\, h(t_0)^{-1} \cdot \exp \int_{t_0}^{t} c\lambda(s)\, ds.$$

For $n > 0$

$$W(u) = \int^u \frac{dz}{z^{n+1}} = -\frac{u^{-n}}{n}, \qquad u > 0$$

and

$$W^{-1}(v) = (-nv)^{-1/n}, \qquad v < 0.$$

Then for y_0 sufficiently small we have for all $t \geqslant t_0$ the following inequalities:

$$\begin{aligned} &|y(t, t_0, y_0)| \\ &\quad \leqslant h(t)\, h(t_0)^{-1} \left[-n \frac{-(c\,|y_0|)^{-n}}{n} - n[h(t_0)^{-1}]^n \int_{t_0}^t c\lambda(s)\, h^n(s)\, ds \right]^{-1/n} \\ &\quad = c\,|y_0|\, h(t)\, h(t_0)^{-1} \left[1 - n(c\,|y_0| h(t_0)^{-1})^n \int_{t_0}^t \lambda(s)\, h^n(s)\, ds \right]^{-1/n} \\ &\quad \leqslant 2c\,|y_0|\, h(t)\, h(t_0)^{-1}. \end{aligned}$$

It is important to note that Theorems 3 and 4 are also valid for $h(t) \equiv 1$ and in this case condition (2) becomes

$$|x(t, t_0, x_0)| \leqslant c\,|x_0|, \qquad (t \geqslant t_0 \geqslant a), \tag{2'}$$

which represents the uniform Lipschitz stability [8]:

THEOREM 5. *Assume that*

(H_0) *For x_0 small enough, there exists a positive constant c such that*

$$|\phi(t, t_0, x_0)| \leqslant c \qquad (t \geqslant t_0 \geqslant a).$$

(H_2) *There is a continuous and nonnegative function λ such that for $(t, y) \in I_a \times D$,*

$$|g(t, y)| \leqslant \lambda(t)\, \omega(|y|), \qquad \lambda \in L_1(I_a),$$

where ω is a continuous, positive, and nondecreasing function which satisfies (21).

Then, (T_1) *of Theorem* 3 *holds and so does*

(T_2) *System* (3) *is uniformly stable.*

(T_3) *For each one of these solutions* y *of* (3), *there is a solution* x *of* (1) *such that for* $t \to \infty$

$$y = x + \tilde{o}(1).$$

If $\omega(u) = u^{n+1}$, $n \geqslant 0$, *by Theorem* 4, (T_2) *becomes* (T_2'). *System* (3) *is also uniformly Lipschitz stable.*

Thus our results extend Theorem 2.14 in Dannan and Elaydi [8].

Finally, we give some asymptotic integration versions of Theorems 1 and 2.

THEOREM 6. *Assume* (H) *and*

$$\limsup_{t \to \infty} h(t) \int_a^t h(s)^{-1} \, ds = M. \tag{13}$$

In addition, suppose that

$$\lim_{t \to \infty} g(t, y) \equiv 0 \tag{22}$$

uniformly in a neighborhood of the origin $y = 0$. *Then*

(T_1') *For* t_0 *sufficiently large and* y_0 *sufficiently small, the solution* $y(t, t_0, y_0)$ *of the perturbed system*

$$y' = f(t, y) + g(t, y) \tag{3}$$

is defined for all $t \geqslant t_0$.

(T_2) *For each one of these solutions* y *of* (3) *there exists a solution* x *of* (1) *such that*

$$y = x + o(1)$$

as $t \to \infty$.

Proof. We obtain (T_1') in a similar way as we did in Theorem 2. The conclusion (T_2) follows from the fact that

$$\lim_{t \to \infty} \int_{t_0}^t \phi(t, s, y(s)) \, g(s, y(s)) \, ds = 0$$

for t_0 sufficiently large since (13) implies (13′).

Note that if g satisfies the inequality

$$|g(t, y)| \leqslant \lambda(t) \, |y|, \qquad \lambda(t) \to 0 \qquad \text{as} \quad t \to \infty$$

then (T_1) is fulfilled for all $t_0 \geqslant a$. This is also true if (22) is replaced by their symmetrical condition

$$\lim_{|y| \to 0} \frac{g(t, y)}{|y|} \equiv 0 \tag{32}$$

uniformly in $t \in I_a$. Moreover, if g satisfies (11) this conclusion is also true if h verifies (12). That is,

THEOREM 7. *Assume that for x_0 small enough,* (6) *is an h-system, where h satisfies* (13), *if either*

(i) *the perturbation g satisfies* (23), *or*

(ii) *h verifies* (12) *and the perturbation g satisfies* (11).

Then (T_1) *of Theorem* 3 *and* (T_2) *of Theorem* 6 *hold.*

Moreover, as it is easy to verify, we can obtain a version, where the perturbation g is the sum $g = g_1 + g_2 + g_3$ with g_1, g_2, g_3 satisfying conditions (8), (9), and (11), respectively. So the results given by Theorem 7 are more precise, via asymptotic formulae, than those given by Theorem 2 and 3 in [10].

4. LINEAR CASE

In this section we will study (1) in the linear case, i.e.,

$$f(t, x) = A(t)\, x,$$

where $A(t)$ is a continuous $m \times m$ matrix on I_a. We will give the form of some theorems of the last section in this case and some particular results valid only where (1) is linear. First, by Theorem 3, we obtain

THEOREM 8. *Assume* (H_1) *and* (H_3) *of Theorem* 3 *and assume also*: (H^*) *One (and then all of them) fundamental matrix ϕ of the system*

$$x' = A(t)\, x \tag{25}$$

satisfies

$$|\phi(t)\, \phi^{-1}(s)| \leqslant ch(t)\, h(s)^{-1} \qquad (t \geqslant s \geqslant a). \tag{26}$$

Then, (T_1) *and* (T_2) *of Theorem* 3 *hold and so does* (T_3). *For each one of these solutions of*

$$y' = A(t)\, y + g(t, y) \tag{27}$$

there exists $z_0 \in \mathbb{R}^m$ *such that for* $t \to \infty$,

$$y = \phi z_0 + h \cdot \tilde{o}(1). \tag{28}$$

Remark 3. As we will see in Section (5), there are linear h-systems (25) for which the following stronger condition holds:

$$|\phi(t)| \leqslant c_1 h(t), \qquad |\phi^{-1}(t)| \leqslant c_2 h(t)^{-1}, \qquad t \geqslant a; \quad c_1, c_2 \text{ constants.} \tag{26'}$$

We call them strong h-systems and we get a more precise version than (28) for them,

$$y = \phi[z_0 + o(1)], \tag{29}$$

because in this case the limit of

$$\int_{t_0}^{t} \phi^{-1}(s)\, g(s, y(s))\, ds$$

exists as $t \to \infty$.

A more precise asymptotic formula is given in the following theorem.

THEOREM 9. *Let* ω *be a continuous, positive, and nondecreasing function on* $[0, \infty)$ *such that* ω *satisfies* (21). *Assume that for one fundamental matrix* ϕ *of* (25) *we have*

$$|\phi^{-1}(t)\, g(t, \phi(t)\, z)| \leqslant \lambda(t)\, \omega(|z|), \qquad \lambda \in L_1(I_a),$$

for $t \in I_a$ *and* $z \in \mathbb{R}^m$.

Then (T_1) *of Theorem* 3 *holds and so does* (T_4). *For any solution* y *as in* (T_1), *there exists* $z_0 \in \mathbb{R}^m$ *such that*

$$y = \phi\left[z_0 + O\left(\int_t^{\infty} \lambda(s)\, ds\right)\right] \tag{30}$$

as $t \to \infty$.

Proof. Making $y = \phi z$ in (27), we obtain

$$z(t) = \phi^{-1}(t_0)\, y_0 + \int_{t_0}^{t} \phi^{-1}(s)\, g(s, \phi(s)\, z(s))\, ds.$$

So,

$$|z(t)| \leqslant |\phi^{-1}(t_0)\, y_0| + \int_{t_0}^{t} \lambda(s)\, \omega(|z(s)|)\, ds$$

and, by Lemma 1,

$$|z(t)| \leqslant W^{-1}\left(W(|\phi^{-1}(t_0)\, y_0|) + \int_{t_0}^{t} \lambda(s)\, ds\right),$$

it follows (T_1) of Theorem 3. Moreover, for any $t_0 \geqslant a$ and y_0 sufficiently small, there exists a positive constant K such that

$$|z(t)| \leqslant K$$

for all $t \geqslant t_0$. Then, for $t \geqslant t_0$,

$$|y(t, t_0, y_0)| \leqslant K\, |\phi(t)|.$$

Moreover, $\phi^{-1}(t)\, g(t, y(t)) \in L_1(I_a)$ and

$$\left|\int_t^\infty \phi^{-1}(s)\, g(s, y(s))\, ds\right| = \left|\int_t^\infty \phi^{-1}(s)\, g(s, \phi(s)\, z(s))\, ds\right| \leqslant K \int_t^\infty \lambda(s)\, ds.$$

So, for each one of these solutions y of (27) the function

$$x(t) = y(t) + \int_t^\infty \phi(t)\, \phi^{-1}(s)\, g(s, y(s))\, ds$$

is solution of (25) and satisfies

$$\phi^{-1}(t)(x(t) - y(t)) = 0\left(\int_t^\infty \lambda(s)\, ds\right),$$

and then (T_4) holds.

Notice that in Theorem 8 the hypothesis over the nonperturbed system (25) is independent of the perturbation g, but in Theorem 9 that dependence really exists. Also these two theorems differ by the error given in their respective asymptotic formulae: In (28) we get only a $h \cdot \tilde{o}(1)$ and in (30) we get an error of order $\phi \cdot o(1)$.

These differences are notorious when the perturbed system (27) is linear.

COROLLARY 1. *Assume that* (H^*) *and* (H_1) *of Theorem* 8 *are satisfied. Then, for each fundamental matrix* Y *of*

$$y' = (A(t) + B(t))\, y, \qquad B \in L_1(I_a) \tag{31}$$

there exists a constant invertible matrix C *such that*

$$Y(t) = \phi(t)\, C + h \cdot \tilde{o}(1)$$

as $t \to \infty$.

COROLLARY 2. *Assume that for one (and then for all of them) fundamental matrix ϕ of (2) we have*

$$\phi^{-1}B\phi \in L_1(I_a).$$

Then, for each fundamental matrix Y of (31) there exists an invertible and constant matrix C such that

$$Y(t) = \phi(t)[C + o(1)].$$

as $t \to \infty$.

The last two Corollaries are different versions of the theorem of N. Levinson [5, p. 92] which does not requiere that $A(t)$ be diagonalizable.

COROLLARY 3. *If the linear system (25) is uniformly stable then for each fundamental matrix Y of the perturbed linear system*

$$y' = (A(t) + B(t))\, y, \qquad B \in L_1(I_a),$$

there exists a fundamental matrix ϕ of (25) such that for $t \to \infty$:

$$Y(t) = \phi(t) + \tilde{o}(1). \tag{32}$$

COROLLARY 4. *If the linear system (25) is strongly stable then the conclusion of Corollary 3 is still valid if we replace (32) by*

$$Y(t) = \phi(t)[I + \tilde{o}(1)].$$

COROLLARY 5. *If all the solutions of the linear system with constant coefficient*

$$x' = Ax, \qquad A \text{ constant},$$

are bounded then for each fundamental matrix Y of the perturbed linear system

$$y' = (A + B(t))\, y, \qquad B \in L_1(I_a),$$

there exists an $m \times m$ constant and invertible matrix C such that for $t \to \infty$,

$$Y(t) = e^{tA}C + \tilde{o}(1).$$

More delicate arguments assure that in this case $\tilde{o}(1)$, the error, is really an $o(1)$ (see [4, Theorem 4.7]). In a next paper, we will obtain general results which will give greated precision in this sense (see [11] and [14]).

With these results one can check that almost all classic results about stability have, via an asymptotic formula, a more precise formulation. For instance:

(i) If (25) is exponentially stable then (27) is of the same type.

(ii) If (25) is strongly stable so is (27).

Also, something similar happens with the famous result due to R. Bellman [1, Theorem 6, p. 43], where we can obtain the asymptotic formula of Corollary 4.

In the next section, we will give other examples showing the variety of equations which can be studied via our results.

5. Examples and Applications

In this section we give examples where we apply the results already obtained, and we study, at the end of the section, the Emden Fowler's equation.

Example. The solutions of the equation

$$y' = -\frac{1}{t}y + \frac{y\cos(y-t^2)\,e^{-(t^2-5y^6)}\ln t}{t+t^3}$$

satisfy, for y_0 sufficiently small,

$$y(t, t_0, y_0) \sim y_0\frac{t_0}{t}, \qquad t \to \infty.$$

Example. Consider the nonlinear equation

$$y' = f(t, y) + g(t, y), \tag{33}$$

where $f(t, x) = \lambda(t)(-x + x^2)$. Here, taking λ a function of the type constructed in Example 1 of Section 2, the nonperturbed equation

$$x' = f(t, x)$$

is an h-system, where

$$h(t) = \exp - \int_a^t \lambda(s)\,ds,$$

which satisfies (12) and (13) and it is not of exponential type.

For x_0 small enough, their variational systems:

$$z' = f_x(t, x(t, t_0, x_0)\, z$$

are also h-systems, i.e., they satisfy (7). So, for any perturbation g of the type considered before, we get that the corresponding solutions of (33) satisfy

$$y(t, t_0, y_0) = x(t, t_0, y_0) + h \cdot \tilde{o}(1), \qquad t \to \infty.$$

Moreover, since h satisfies (12) and (13), we can apply Theorems 6 and 7 to (33).

EXAMPLE. Consider the equation

$$y' = -y^3 + \frac{y^7 e^{-t}}{1+3y^6} - \frac{y^6 - 2y}{(t+ty^2)^2} \sin(ty). \tag{34}$$

Here, the nonperturbed equation

$$x' = -x^3$$

possesses the solutions

$$x(t, t_0, x_0) = x_0(1 + 2x_0^2(t - t_0))^{-1/2}.$$

It is easy to see that

$$|x(t, t_0, x_0)| \leqslant |x_0| \qquad (t \geqslant t_0)$$

and also that there is a constant $K > 0$ such that

$$|\phi(t, t_0, x_0)| = \left| \frac{\partial x(t, t_0, x_0)}{\partial x_0} \right| \leqslant K \qquad (t \geqslant t_0).$$

So, by Theorem 5, Eq. (34) has solutions $y(t, t_0, y_0)$ for $t_0 \geqslant a$ and y_0 small enough such that for $t \to \infty$,

$$y(t, t_0, y_0) = y_0[1 + 2y_0^2(t - t_0)]^{-1/2} + \tilde{o}(1).$$

EXAMPLE. Consider the second-order equation

$$t^2 y'' + ty' + (t^2 - \alpha^2)\, y + y^{n+1} = 0, \qquad n > 2, \quad \alpha \text{ constant}. \tag{35}$$

Here, the nonperturbed equation is the Bessel equation

$$t^2 u'' + tu' + (t^2 - \alpha^2)\, u = 0, \qquad \alpha \text{ constant}. \tag{36}$$

The change of variable $v = t^{1/2}u$, transforms the equation of Bessel in the following one

$$v'' + \left(1 + \frac{1/2 - \alpha^2}{t^2}\right) v = 0$$

which can be written as a linear system of second order

$$x' = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1/4 - \alpha^2}{t^2} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] x.$$

By Corollary 4, this system has a fundamental matrix ϕ such that for $t \to \infty$,

$$\phi(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} (I + o(1)).$$

Therefore, the analogous system corresponding to the Bessel's equation (36) has a fundamental matrix $U(t)$ such that for $t \to \infty$,

$$U(t) = t^{-1/2} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} (I + o(1)),$$

concluding that the corresponding system to the Bessel's equation is a strongly h-system with $h(t) = t^{-1/2}$. So by Theorem 8 and Remark 3 Eq. (35) and the equation

$$t^2 y'' + t y' + (t^2 - \alpha^2) y + g(t, y) = 0,$$

where

$$|g(t, y)| \leqslant \lambda(t)\, |y|^{n+1}, \qquad \lambda(t)\, t^{-n/2} \in L_1(I_a) \qquad (a > 0),$$

$n \geqslant 0$, have solutions y such that for $t \to \infty$,

$$y \sim c_1 J_\alpha + c_2 N_\alpha,$$
$$y' \sim c_1 J'_\alpha + c_2 N'_\alpha,$$

where c_1 and c_2 are constant and J_α and N_α are the Bessel's and Neuman's functions of order α, respectively.

The same conclusion is valid if g satisfies the condition (H_2) of Theorem 3.

EXAMPLE. Consider the second-order linear equation:

$$y'' + (b(t) + c(t))\, y = 0, \qquad c \in L_1(I_a). \tag{37}$$

If the solutions of the nonperturbed equation

$$x'' + b(t)\,x = 0 \tag{38}$$

are bounded (that is, the case if, for instance, $b(t)$ increases to infinity as $t \to \infty$), then by Corollary 2 for each solution y of (37) there exists a solution x of (38) such that for $t \to \infty$,

$$y = x(1 + o(1)), \qquad y' = x'(1 + o(1)). \tag{39}$$

Other conditions, on the other hand, which assure the same result are given in Coppel [6, p. 122] and Pinto [12]. In this last paper, we obtain asymptotic formulae for second-order linear equations which are similar to those that we have obtained in (39).

Formula (39) is a nice result; it is more precise and it includes Theorem 2 given in Bellman [1, p. 112].

In the same tack it would be interesting to obtain the corresponding result under the conditions

$$y'' + (b(t) + c(t))\,y = 0, \qquad c(t) \to 0, \quad t \to \infty, \qquad \text{and} \qquad c' \in L_1(I_a).$$

(see Bellman [1, Theorem 1, p. 112]).

We end this section studying the Emden–Fowler equation which appears in astrophysics and atomic physics

$$(t^\delta u')' \pm t^\gamma u^n = 0, \qquad \delta, \gamma, n \text{ constants}.$$

It can be transformed (see Bellman [1, p. 143]) into

$$u'' \pm t^\sigma u^n = 0.$$

So, it can be written as

$$z' = Az \pm k(t, z), \tag{40}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad k(t, z) = t^\sigma z_2^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} u' \\ u \end{pmatrix}.$$

Making the change of variable $z = \operatorname{diag}\{1, t\}\, e^A y$ one has

$$y' = t^{-1} \Lambda y \pm g(t, y), \tag{41}$$

where

$$\Lambda = \operatorname{diag}\{0, -1\}, \qquad g(t, y) = e^{-A} \operatorname{diag}\{1, t^{-1}\}\, k(t, \operatorname{diag}\{1, t\}\, e^A y).$$

Since $k_1(t, z) = t^\sigma z_2^n$, we obtain

$$|g(t, y)| \leqslant \alpha_1 \left| \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot k_1(t, \operatorname{diag}\{1, t\}\, e^A y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$$

$$= \alpha_1 \, |k_1(t, \operatorname{diag}\{1, t\}\, e^A y)|$$

$$\leqslant \alpha_1 t^\sigma \cdot t^n \cdot \alpha_2 \, |y_2|^n \leqslant \alpha t^{\sigma+n} \, |y|^n.$$

Then, for $\sigma + n < -1$, we can apply Theorem 5 and therefore for each solution y of (41) with small initial conditions there exists $z_0 \in \mathbb{R}^2$ such that for $t \to \infty$,

$$y(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} z_0 + \tilde{o}(1).$$

So,

$$z(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} z_0 + \tilde{o}(1) \right],$$

i.e.,

$$u' = z_1 = z_1^0 + \tilde{o}(1) \qquad y = u = z_2 = z_2^0 + \tilde{o}(1) + t(z_1^0 + \tilde{o}(1)).$$

Moreover, by definition of $\tilde{o}(1)$, we can restate these formulae by writing

$$u = c_1 + o(1) + t(c_2 + o(1)) \qquad \text{and} \qquad u' = c_2 + o(1),$$

where c_1 and c_2 are constants. So, $u^n = c_1^n + o(1) + t^n(c_2^n + o(1))$. Then,

$$u'(t) = -\int_t^\infty s^\sigma u^n(s)\, ds = \frac{t^{\sigma+n+1}}{\sigma+n+1} (c_2^n + o(1)) + \frac{t^{\sigma+1}}{\sigma+1} (c_1^n + o(1)) + c_2$$

and

$$\begin{aligned} u(t) &= -\int_t^\infty u'(s)\, ds \\ &= \frac{t^{\sigma+n+2}}{(\sigma+n+1)(\sigma+n+2)} (c_2^n + o(1)) + \frac{t^{\sigma+2}}{(\sigma+1)(\sigma+2)} (c_1^n + o(1)) \\ &\quad + c_1 + tc_2 \end{aligned}$$

and this is the formula that Emden found. Formula (2) in Theorem 1 of Bellman [1], corresponding to the case $u'(t) \to \infty$ as $t \to \infty$, can be obtained by transforming (40) as Bellman did there.

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